NON-DEGENERATE JUMP OF MILNOR NUMBERS OF SURFACE SINGULARITIES

SZYMON BRZOSTOWSKI, TADEUSZ KRASIŃSKI AND JUSTYNA WALEWSKA

Abstract. The jump of the Milnor number of an isolated singularity f_0 is the minimal non-zero difference between the Milnor numbers of f_0 and one of its deformations (f_s). We give a formula for the jump in some class of surface singularities in the case deformations are non-degenerate.

1. Introduction

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an *(isolated) singularity*, i.e. let f_0 be a germ at 0 of a holomorphic function having an isolated critical point at $0 \in \mathbb{C}^n$, and $0 \in \mathbb{C}$ as the corresponding critical value. More specifically, there exists a representative $\hat{f}_0 : U \to \mathbb{C}$ of f_0 holomorphic in an open neighborhood U of the point $0 \in \mathbb{C}^n$ such that:

- $\hat{f}_0(0) = 0$,
- $\nabla \hat{f}_0(0) = 0$,
- $\nabla \hat{f}_0(z) \neq 0$ for $z \in U \setminus \{0\}$,

where for a holomorphic function f we put $\nabla f := (\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$.

In the sequel we will identify germs of functions with their representatives or the corresponding convergent power series. The ring of germs of holomorphic functions of n variables will be denoted by $\mathcal{O}_n$.

A *deformation of the singularity* f_0 is a germ of a holomorphic function $f = f(s,z) : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that:

- $f(0,z) = f_0(z)$,
- $f(s,0) = 0$,

The deformation $f(s,z)$ of the singularity f_0 will also be treated as a family (f_s) of germs, putting $f_s(z) := f(s,z)$. Since f_0 is an isolated singularity, f_s has also isolated singularities near the origin, for sufficiently small s [GLS07, Theorem 2.6 in Chap. I].

Remark. *Notice that in the deformation (f_s) there can occur in particular* smooth *germs, that is germs satisfying $\nabla f_s(0) \neq 0$. In this context, the symbol ∇f_s will always denote $\nabla_z f_s(z)$.*

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By the above assumptions it follows that, for every sufficiently small s, one can define a (finite) number μ_s as the Milnor number of f_s, namely

$$\mu_s := \mu(f_s) = \dim_{\mathbb{C}} \mathcal{O}_n/(\nabla f_s) = \mu\left(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n}\right),$$

where the symbol $\mu(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n})$ denotes intersection multiplicity of the ideal $(\frac{\partial f}{\partial z_1}, \dots, \frac{\partial f}{\partial z_n})\mathcal{O}_n$ in $\mathcal{O}_n$.

Since the Milnor number is upper semi-continuous in the Zariski topology in families of singularities [GLS07, Theorem 2.57 in Chap. II], there exists an open neighborhood S of the point $0 \in \mathbb{C}$ such that

- $\mu_s = \text{const.}$ for $s \in S \setminus \{0\}$,
- $\mu_0 \geqslant \mu_s$ for $s \in S$.

The (constant) difference $\mu_0 - \mu_s$ for $s \in S \setminus \{0\}$ will be called *the jump of the deformation* (f_s) and denoted by $\lambda((f_s))$. The smallest nonzero value among all the jumps of deformations of the singularity f_0 (such a value exists because one can always consider a deformation of f_0 built of smooth germs and then for it it is $\mu_s = 0$; cf. Remark 1) will be called *the jump (of the Milnor number) of the singularity* f_0 and denoted by $\lambda(f_0)$.

The first general result concerning the jump was S. Gusein-Zade's [GZ93], who proved that there exist singularities f_0 for which $\lambda(f_0) > 1$ and that for irreducible plane curve singularities it holds $\lambda(f_0) = 1$. In [BK14] the authors proved that $\lambda(f_0)$ is not a topological invariant of f_0 but it is an invariant of the stable equivalence. The computation of $\lambda(f_0)$ for a specific reducible singularity (or for a class of reducible singularities) is not an easy task. It is related to the problem of adjacency of classes of singularities. Only for a few classess of singularities we know the exact value of $\lambda(f_0)$. For plane curve singularities ($n = 2$) we have (see [AGZV85] for terminology):

- for the one-modal family of singularities in the X_9 class, that is singularities of the form
$$f_0^a(x,y) := x^4 + y^4 + ax^2y^2, \quad a \in \mathbb{C}, \quad a^2 \neq 4,$$
we have $\lambda(f_0^a) = 2$ ([BK14]),
- for the two-modal family of singularities in the $W_{1,0}$ class, that is singularities of the form
$$f_0^{a,b}(x,y) := x^4 + y^6 + (a + by)x^2y^3, \quad a,b \in \mathbb{C}, \quad a^2 \neq 4,$$
we have
$$\lambda(f_0^{a,b}) = \begin{cases} 1, & \text{if } a = 0 \text{ ([BK14])} \\ \geqslant 2, & \text{for generic } a,b \text{ ([GZ93])}, \end{cases}$$
- for specific homogenous singularities $f_0^d(x,y) := x^d + y^d$, $d \geqslant 2$, we have $\lambda(f_0^d) = \left[\frac{d}{2}\right]$ ([BKW14]),
- for homogeneous singularities of degree d with generic coefficients f_0 we have $\lambda(f_0) < \left[\frac{d}{2}\right]$ ([BKW14])

In the present paper we consider a weaker problem: *compute the jump* $\lambda^{\mathrm{nd}}(f_0)$ *of* f_0 *over all non-degenerate deformations of* f_0 (i.e. the f_s in the deformations (f_s) of f_0 are non-degenerate singularities). Clearly, we always have $\lambda(f_0) \leqslant \lambda^{\mathrm{nd}}(f_0)$. Up to now, this problem has been studied only for plane curve singularities

- A. Bodin ([Bod07]) gave a formula for $\lambda^{\mathrm{nd}}(f_0)$ for f_0 convenient with its Newton polygon reduced to one segment,
- J. Walewska in [Wal13] generalized Bodin's results to the non-convenient case,
- the authors ([BKW14]) calculated all possible Milnor numbers of all non-degenerate deformations of homogenous singularities,
- J. Walewska ([Wal10]) proved that the *second non-degenerate jump of* f_0 satisfying Bodin's assumptions is equal to 1.

In this paper we want to pass to surface singularities ($n = 3$). We give a formula (more precisely: a simple algorithm) for $\lambda^{\mathrm{nd}}(f_0)$ in the case where f_0 is non-degenerate, convenient and has its Newton diagram reduced to one triangle, (see Figure 1) i.e. f_0 of the form

$$f_0(x,y,z) = ax^p + by^q + cz^r + \ldots \quad (p,q,r \geqslant 2,\ \ abc \neq 0).$$

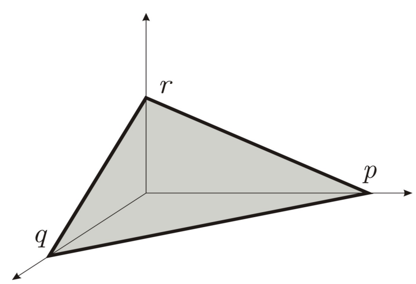

FIGURE 1. The Newton diagram of $f_0(x,y,z) = ax^p + by^q + cz^r + \ldots$

Moreover, for simplicity reasons, we will only consider the case of p, q, r being pairwise coprime integers. The general case of arbitrary p, q, r will be the topic of a next paper.

2. Non-degenerate singularities

In this Section we recall the notion of non-degenerate singularities. We restrict ourselves to surface singularities. All notions can easily be generalized to higher dimensions. Let $f_0(x,y,z) := \sum_{i,j,k\in\mathbb{N}} a_{ijk}x^i y^j z^k$, be a singularity. Let $\mathrm{supp}(f_0) := \{(i,j,k) \in \mathbb{N}^3 : a_{ijk} \neq 0\}$ be the *support of* f_0. The *Newton polyhedron* $\Gamma_+(f_0)$ *of* f_0 is the convex hull of the set

$$\bigcup_{(i,j,k)\in\mathrm{supp}(f_0)} (i,j,k) + \mathbb{R}^3_+,$$

where $\mathbb{R}^3_+$ is the closed octant of $\mathbb{R}^3$ consisting of points with nonnegative coordinates. The boundary (in $\mathbb{R}^3$) of $\Gamma_+(f_0)$ is an unbounded polyhedron with a finite

number of 2-dimensional faces, which are (not necessarily compact) polygons. The singularity f_0 is called *convenient* if $\Gamma_+(f_0)$ has some points in common with all three coordinate axes in $\mathbb{R}^3$. The set of compact faces (of all dimensions) of $\Gamma_+(f_0)$ constitutes the *Newton diagram of* f_0 and is denoted by $\Gamma(f_0)$. For each face $S \in \Gamma(f_0)$ we define a weighted homogeneous polynomial

$$(f_0)_S := \sum_{(i,j,k)\in S} a_{ijk} x^i y^j z^k.$$

We call the singularity f_0 *non-degenerate on* $S \in \Gamma(f_0)$ if the system of equations

$$\frac{\partial (f_0)_S}{\partial x}(x,y,z) = 0, \quad \frac{\partial (f_0)_S}{\partial y}(x,y,z) = 0, \quad \frac{\partial (f_0)_S}{\partial z}(x,y,z) = 0$$

has no solutions in $(\mathbb{C}^*)^3$; f_0 is *non-degenerate* (*in the Kouchnirenko sense*) if f_0 is non-degenerate on every face $S \in \Gamma(f_0)$.

Assume now that f_0 is convenient. We introduce the following notation:

- $\Gamma_-(f_0)$ – the compact polyhedron bounded by $\Gamma(f_0)$ and the three coordinate planes (labeled in a self-explanatory way as OXY, OXZ, OYZ); in other words, $\Gamma_-(f_0) := \overline{\mathbb{R}^3_+ \setminus \Gamma_+(f_0)}$,
- V – the volume of $\Gamma_-(f_0)$,
- P_1, P_2, P_3 – the areas of the two-dimensional faces of $\Gamma_-(f_0)$ lying in the planes OXY, OXZ, OYZ, respectively; e.g. P_1 is the area of the set $\Gamma_-(f_0) \cap$ OXY,
- W_1, W_2, W_3 – the lengths of the edges (= one-dimensional faces) of $\Gamma_-(f_0)$ lying in the axes OX, OY, OZ, respectively (see Figure 2).

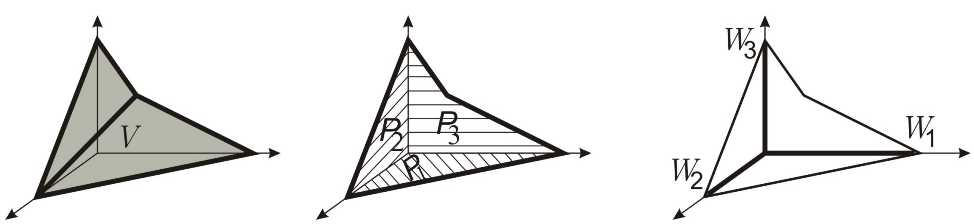

FIGURE 2. Geometric meaning of volume V, areas P_i and lengths W_j.

We define the *Newton number* $\nu(f_0)$ of f_0 by

$$(\circ) \qquad \nu(f_0) := 3!V - 2!(P_1 + P_2 + P_3) + 1!(W_1 + W_2 + W_3) - 1.$$

The importance of $\nu(f_0)$ has its source in the celebrated *Kouchnirenko theorem*:

Theorem 2.1 ([Kou76]). *If f_0 is a convenient singularity, then*

(1) $\mu(f_0) \geqslant \nu(f_0)$,
(2) if f_0 is non-degenerate then $\mu(f_0) = \nu(f_0)$.

Remark 1. *The Kouchnirenko theorem is true in any dimension [Kou76].*

3. Non-degenerate jump of Milnor numbers of singularities

Let $f_0 \in \mathcal{O}_3$ be a singularity. A deformation (f_s) of f_0 is called *non-degenerate* if f_s is non-degenerate for $s \neq 0$. The set of all non-degenerate deformations of the singularity f_0 will be denoted by $\mathcal{D}^{\mathrm{nd}}(f_0)$. *Non-degenerate jump* $\lambda^{\mathrm{nd}}(f_0)$ *of the singularity* f_0 is the minimal of non-zero jumps over all non-degenerate deformations of f_0, which means

$$\lambda^{\mathrm{nd}}(f_0) := \min_{(f_s)\in\mathcal{D}_0^{\mathrm{nd}}(f_0)} \lambda((f_s)),$$

where by $\mathcal{D}_0^{\mathrm{nd}}(f_0)$ we denote all the non-degenerate deformations (f_s) of f_0 for which $\lambda((f_s)) \neq 0$.

Obviously

Proposition 3.1. *For each singularity f_0 we have the inequality*

$$\lambda(f_0) \leqslant \lambda^{\mathrm{nd}}(f_0).$$

In investigations concerning $\lambda^{\mathrm{nd}}(f_0)$ we may restrict our attention to non-degenerate f_0 because the non-degenerate jump for degenerate singularities can be found using the proposition below (cf. [Bod07, Lemma 5]). Let f_0^{nd} denote any non-degenerate singularity for which $\Gamma(f_0) = \Gamma(f_0^{\mathrm{nd}})$. Such singularities always exist.

Proposition 3.2. *If f_0 is degenerate then*

$$\lambda^{\mathrm{nd}}(f_0) = \begin{cases} \mu(f_0) - \mu(f_0^{\mathrm{nd}}), & \text{if } \mu(f_0) - \mu(f_0^{\mathrm{nd}}) > 0 \\ \lambda^{\mathrm{nd}}(f_0^{\mathrm{nd}}), & \text{if } \mu(f_0) - \mu(f_0^{\mathrm{nd}}) = 0 \end{cases}.$$

Proof. This follows from the fact that a generic small perturbation of coefficients of these monomials of f_0 which correspond to points belonging to $\bigcup \Gamma(f_0)$ (which are finite in number) give us non-degenerate singularities with the same Newton polyhedron as f_0. □

Remark 2. *By the Płoski theorem ([Pło90, Lemma 2.2], [Pło99, Theorem 1.1]), for degenerate plane curve singularities ($n = 2$) the second possibility in Proposition 3.2 is excluded.*

A crucial rôle in the search for the formula for $\lambda^{\mathrm{nd}}(f_0)$ will be played by the monotonicity of the Newton number with respect to the Newton polyhedron. Namely, J. Gwoździewicz [Gwo08] and M. Furuya [Fur04] proved:

Theorem 3.3 (Monotonicity Theorem)**.** *Let $f_0, \tilde{f}_0 \in \mathcal{O}_n$ be two convenient singularities such that $\Gamma_+(f_0) \subset \Gamma_+(\tilde{f}_0)$. Then $\nu(f_0) \geqslant \nu(\tilde{f}_0)$.*

By this theorem the problem of calculation of $\lambda^{\mathrm{nd}}(f_0)$ can be reduced to a purely combinatorial one. Namely, we define specific deformations of a convenient and non-degenerate singularity $f_0 \in \mathcal{O}_n$. Denote by J the set of integer points $\boldsymbol{i} = (i_1, \dots, i_n) \neq 0$ lying in the closed domain bounded by coordinate hyperplanes $\{z_i = 0\}$ and the Newton diagram; in other words $J := \Gamma_-(f_0) \cap \mathbb{Z}^n$. Obviously, J is a finite set. For $\boldsymbol{i} = (i_1, \dots, i_n) \in J$ we define the deformation $(f_s^{\boldsymbol{i}})_{s\in\mathbb{C}}$ of f_0 by the formula

$$f_s^{\boldsymbol{i}}(z_1, \dots, z_n) := f_0(z_1, \dots, z_n) + s z_1^{i_1} \dots z_n^{i_n}.$$

Proposition 3.4. *For every $\boldsymbol{i} \in J$ the deformation $(f_s^{\boldsymbol{i}})$ of f_0 is convenient and non-degenerate for all sufficiently small $|s|$.*

Proof. See [Kou76] or [Oka79, Appendix].

$\square$

Combining the Monotonicity Theorem with the above proposition we reach the conclusion that in order to find $\lambda^{\mathrm{nd}}(f_0)$ it is enough to consider only the non-degenerate deformations of the type $(f_s^{\boldsymbol{i}})$.

Theorem 3.5. *If f_0 is a convenient and non-degenerate singularity, then*

$$\lambda^{\mathrm{nd}}(f_0) = \min_{\boldsymbol{i} \in J_0} \lambda((f_s^{\boldsymbol{i}}))$$

where $J_0 \subset J$ is the set of these $\boldsymbol{i} \in J$ for which $\lambda^{\mathrm{nd}}((f_s^{\boldsymbol{i}})) > 0$.

Proof. By the Kouchnirenko theorem it suffices to consider non-degenerate deformations of f_0 of the form

$$(\ast) \qquad f_s(z_1, \dots, z_n) = f_0(z_1, \dots, z_n) + \sum_{\boldsymbol{i} \in J} a_{\boldsymbol{i}}(s) z^{\boldsymbol{i}},$$

where $a_{\boldsymbol{i}}(s)$ are holomorphic at $0 \in \mathbb{C}$ and $a_{\boldsymbol{i}}(0) = 0$. Then by the Monotonicity Theorem we may restrict the scope of deformations $(\ast)$ to deformations with only one term added i.e. the deformations $(f_s^{\boldsymbol{i}})$ for $\boldsymbol{i} \in J_0$. $\square$

Corollary 3.6. *If f_0 and $\tilde{f}_0$ are non-degenerate and convenient singularities and $\Gamma(f_0) = \Gamma(\tilde{f}_0)$ then $\lambda^{\mathrm{nd}}(f_0) = \lambda^{\mathrm{nd}}(\tilde{f}_0)$.*

4. An algorithm for $\lambda^{\mathrm{nd}}(f_0)$ in the case of one face Newton diagram of surface singularities

In this Section we give a simple algorithm for calculating $\lambda^{\mathrm{nd}}(f_0)$ provided that $f_0 \in \mathcal{O}_3$ is a convenient and non-degenerate singularity with one two-dimensional face of its Newton diagram. Let p, q, r be the first (i.e. nearest to the origin) points of $\Gamma_+(f_0)$ lying on the axes OX, OY and OZ, respectively. Then by Corollary 3.6 we may assume that

$$f_0(x, y, z) = x^p + y^q + z^r, \qquad p, q, r \geqslant 2.$$

By formula $(\circ)$ we have $\mu(f_0) = (p-1)(q-1)(r-1)$. Moreover, without loss of generality we may also assume that

$$(\dagger) \qquad p \geqslant q \geqslant r.$$

Additionally, we demand that p, q, r are pairwise coprime

$$(\ast\ast) \qquad \mathrm{GCD}(p, q) = \mathrm{GCD}(p, r) = \mathrm{GCD}(q, r) = 1.$$

By Theorem 3.5 we have to compare the jumps of deformations $(f_s^{\boldsymbol{i}})_{s \in \mathbb{C}}$, where $\boldsymbol{i} \in J$, i.e. $\boldsymbol{i}$ are integer points lying in the octant of $\mathbb{R}^3$ under the triangle with vertices $(p, 0, 0)$, $(0, q, 0)$, $(0, 0, r)$ (see Figure 1).

I. First we consider points in J lying on the axes. Using formula ($\circ$) and assumption ($\dagger$) we easily check that the axes-jump is realized by the deformation $(f_s^{(p-1,0,0)})$, i.e.

$$f_s^{(p-1,0,0)}(x,y,z) = x^p + y^q + z^r + sx^{p-1},$$

and the jump is equal to $(q-1)(r-1)$.

II. Now we consider points in J lying in coordinate planes. By the results of Bodin [Bod07] and Walewska [Wal10] we easily check that the minimal jumps on respective planes are realized by

i. the deformation $(f^{(b_1,q-a_1,0)})$, where $a_1, b_1 \in \mathbb{Z}$ are such that $a_1p - b_1q = 1$ and $0 < a_1 < q$, $b_1 > 0$; this delivers the OXY-jump equal to $(r-1)$,

ii. the deformation $(f_s^{(0,b_2,r-a_2)})$, where $a_2, b_2 \in \mathbb{Z}$ are such that $a_2q - b_2r = 1$ and $0 < a_2 < r$, $b_2 > 0$; this delivers the OYZ-jump equal to $(p-1)$,

iii. the deformation $(f_s^{(b_3,0,p-a_3)})$, where $a_3, b_3 \in \mathbb{Z}$ are such that $a_3p - b_3r = 1$ and $0 < a_3 < p$, $b_3 > 0$; this delivers the OXZ-jump equal to $(q-1)$.

The above considerations imply that the jump realized by the points lying either in coordinate planes or on axes is equal to $(r-1)$.

III. Let us pass to the deformations $(f_s^{\boldsymbol{i}})$ for which the point $\boldsymbol{i}$ lies in the interior of the tetrahedron with vertices $(0,0,0)$, $(p,0,0)$, $(0,q,0)$, $(0,0,r)$. Any such point (α,β,γ) satisfies the conditions:

(A) $0 < \alpha < p$, $0 < \beta < q$, $0 < \gamma < r$,

(B) $\dfrac{\alpha}{p} + \dfrac{\beta}{q} + \dfrac{\gamma}{r} < 1$ or equivalently $\alpha qr + \beta pr + \gamma pq < pqr$.

Moreover, the jump of the deformation $(f_s^{(\alpha,\beta,\gamma)})$ is equal to 6 times the volume of the tetrahedron with vertices $(p,0,0)$, $(0,q,0)$, $(0,0,r)$, (α,β,γ) i.e.

$$pqr - \alpha qr - \beta pr - \gamma pq.$$

Thus, we have reduced our original problem to a number theoretic one.

Problem. *Given pairwise coprime integers $p > q > r$ greater than 1. Find positive integers α, β, γ satisfying (A) and (B) for which the expression $pqr - \alpha qr - \beta pr - \gamma pq$ attains its positive minimum.*

In order to solve it, first notice that $\mathrm{GCD}(qr, pr, pq) = 1$. Consequently, there are integers a, b, c such that

$$aqr + bpr + cpq = 1. \tag{$\ddagger$}$$

They can be obtained by the Euclid algorithm using the well-known associativity law: *for any integers u, v, w we have* $\mathrm{GCD}(u,v,w) = \mathrm{GCD}(\mathrm{GCD}(u,v),w)$. Notice that in any identity of the type ($\ddagger$) it holds $abc \neq 0$. If we write $a = a'p + a''$, $0 \leqslant a'' < p$, then, by abuse of notation, we obtain yet another identity $aqr + bpr + cpq = 1$, but now $0 < a < p$. Next, we write $b = b'q - b''$, $0 < b'' < q$, and we use it to obtain a similar identity $aqr - bpr + cpq = 1$ in which $0 < a < p$ and $0 < b < q$. Notice that then $0 < |c| < r$. In fact, $|cpq| = |1 - aqr + bpr| \leqslant 1 + r|bp - aq| \leqslant 1 + r(pq - p - q) = pqr - pr - qr + 1 < pqr$. Thus, finally we have obtained the identity

$$(\square) \qquad aqr - bpr + cpq = 1, \text{ where } 0 < a < p,\ 0 < b < q,\ 0 < |c| < r.$$

Now we consider two cases:

(1) $c < 0$. Then the triple $\alpha = p - a$, $\beta = b$, $\gamma = -c$ is the solution that we seek for. In fact, α, β, γ clearly satisfy (A), moreover $pqr - \alpha qr - \beta pr - \gamma pq = aqr - bpr + cpq = 1$. This is the optimal value one can hope for, so the Problem is solved in this case. Hence $\lambda^{\mathrm{nd}}(f_0) = 1$ and the deformation $(f_s^{p-a,b,-c})$ realizes the jump 1.

(2) $c > 0$. Under this condition, we claim that there is no point (α, β, γ) satisfying both (A) and (B) and for which the minimum in the Problem is equal to 1. In fact, if there existed such a point, then from the relation $pqr - \alpha qr - \beta pr - \gamma pq = 1$ we would get $(p - \alpha)qr - \beta pr - \gamma pq = 1$, which together with $(\square)$ would imply that $(p - (\alpha + a))qr = (\beta - b)pr + (\gamma + c)pq$. But since $\mathrm{GCD}(p, r) = \mathrm{GCD}(p, q) = 1$ and $|p - (\alpha + a)| < p$, this is only possible when $\alpha = p - a$. Hence, we would get $(\beta - b)r + (\gamma + c)q = 0$. Similarly, since $\mathrm{GCD}(r, q) = 1$ and $|\beta - b| < q$, we would obtain $\beta = b$ and consequently $\gamma = -c < 0$, contradictory to (A).

The above observation means that in case (2) we must further continue our search for α, β, γ solving the Problem. Accordingly, we repeat the above reasoning for the identity

$$aqr + bpr + cpq = 2,$$

and so on up to

$$aqr + bpr + cpq = r - 2.$$

If in one of the above steps we find integers a, b, c such that

$$aqr + bpr + cpq = i_0,$$

where $1 \leqslant i_0 \leqslant r - 2$, $0 < a < p$, $-q < b < 0$ and $-r < c < 0$, then we stop the procedure and the triple $\alpha = p - a$, $\beta = -b$, $\gamma = -c$ solves the Problem with minimum equal to i_0. Hence, $\lambda^{\mathrm{nd}}(f_0) = i_0$ and the deformation $(f_s^{(p-a,-b,-c)})$ realizes this jump.

If the above search fails, we conclude that $\lambda^{\mathrm{nd}}(f_0) = r - 1$ because the deformation $(f_s^{(b_1, q-a_1, 0)})$, where $a_1 p - b_1 q = 1$, $0 < a_1 < q$, $0 < b_1$, realizes this jump.

We may sum up the above considerations in the following theorem.

Theorem 4.1. *Let $f_0 \in \mathcal{O}_3$ be a convenient and non-degenerate singularity with only one two-dimensional face in its Newton diagram. Assume that the vertices $(p, 0, 0)$, $(0, q, 0)$, $(0, 0, r)$ of this face are such that $p \geqslant q \geqslant r \geqslant 2$ and the numbers p, q, r are pairwise coprime. Then*

$$\lambda^{\mathrm{nd}}(f_0) = \begin{cases} i_0 & \begin{array}{l} \textit{if there exist integers } a, b, c \textit{ such that} \\ aqr + bpr + cpq = i_0,\ 1 \leqslant i_0 \leqslant r - 2, \\ 0 < a < p,\ 0 < -b < q,\ 0 < -c < r,\ i_0 - \textit{minimal}, \end{array} \\ & \\ r - 1 & \textit{otherwise.} \end{cases}$$

Moreover, i_0 can be found algorithmically using only Euclid's algorithm.

Corollary 4.2. *Under the assumptions of Theorem 4.1, if $r = 2$ then $\lambda^{\mathrm{nd}}(f_0) = 1$.*

Example. *For $f_0(x, y, z) := x^{11} + y^6 + z^5$ we have $p = 11$, $q = 6$, $r = 5$ and*

$$\begin{aligned} 7 \cdot qr - 5 \cdot pr + 1 \cdot pq &= 1 \quad \text{– does not satisfy the conditions in the theorem} \\ 3 \cdot qr - 4 \cdot pr + 2 \cdot pq &= 2 \quad \text{– does not satisfy the conditions in the theorem} \\ 10 \cdot qr - 3 \cdot pr - 2 \cdot pq &= 3 \quad \text{– do satisfy the conditions in the theorem.} \end{aligned}$$

Hence, $\lambda^{\mathrm{nd}}(f_0) = 3$. This jump is realized by the deformation $f_s^{(1,3,2)}(x, y, z) := x^{11} + y^6 + z^5 + sxy^3z^2$. The minimal jump realized by the points lying either in coordinate planes or on axces in equal to $r - 1 = 4$.

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